Chapter 5: Relations

8.1 Relations and Their Properties

Binary relations:

- Let A and B be any two sets.
- A **binary relation** R from A to B, written

$$R : A \rightarrow B,$$

is a subset of the Cartesian product $A \times B$.

- The notation $a\ R\ b$ means $(a, b) \in R$.
- The notation $a\ \not R\ b$ means $(a, b) \notin R$.
- If $a\ R\ b$ we may say that a is related to b (by relation R), or a relates to b (under relation R).

Example

Let $R : A \rightarrow B$, and

$A = \{1, 2, 3\}$ represents students,

$B = \{a, b\}$ represents courses.

$A \times B = \{ (1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

If $R = \{(1, a), (1, b)\}$, it means that student 1 registered in courses a and b

Relations can be Represented by:

Let A be the set {1, 2, 3, 4} for which ordered pairs are in the relation

$$R = \{(a, b) \mid a \text{ divides } b\}$$

A- Roaster Notation: List of ordered pairs:

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$$

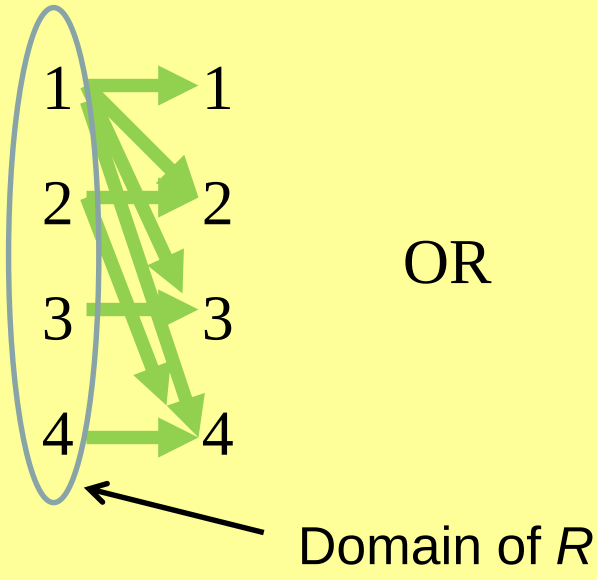
B- Set builder notation:

$$R = \{(a, b) : a \text{ divides } b\}$$

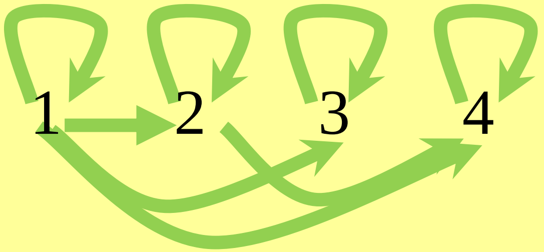
Relations can be Represented by:

C- Graph:

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$



OR



Relations can be Represented by:

D- Table:

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Relations on a Set

- A (binary) relation from a set A to itself is called a **relation on** the set A.

e.g. The "**<**" relation defined as a relation on the set **N** of natural numbers:

$$\text{let } < : \mathbf{N} \to \mathbf{N} :\equiv \{(a, b) \mid a < b\}$$

If $(a, b) \in R$ then $a < b$ means $(a, b) \in <$

e.g. $(1, 2) \in <$.

Relations on a Set

- The **identity relation** I_A on a set A is the set

$$\{(a, a) \mid a \in A\}.$$

e.g. If $A = \{1, 2, 3, 4\}$,

then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

Relations on a Set

Examples:

$$R_1 = \{(a, b) \mid a \le b\}$$

$$R_2 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_3 = \{(a, b) \mid a + b \le 3\}$$

Which of these relations contain each of the following pairs (1, 1), (1, 2), (1, -1)?

(1, 1) is in R_1, R_2, R_3

(1, 2) is in R_1, R_3

(1, -1) is in R_2, R_3

Question

How many relations are there on a set with *n* elements?

Answer:

1. A relation on set *A* is a subset from *A*×*A*.
2. *A* has *n* elements so *A*×*A* has n^2 elements.
3. Number of subsets for n^2 elements is 2^{n^2}, thus there are 2^{n^2} relations on a set with *n* elements.

e.g. If *S* = {*a*, *b*, *c*}, there are $2^{3^2} = 2^9 = 512$ relations.

Properties of Relations
1. Reflexivity and Irreflexivity

A relation R on A is **reflexive** if $(a, a) \in R$ for every element $a \in A$.

e.g. Consider the following relations on {1, 2, 3, 4}

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

Not Reflexive.

$R_2 = \{(1, 1), (2, 1), (2, 2), (3, 3), (3, 4), (4, 4)\}$,

Reflexive.

$R_3 = \{(a, b) \mid a \leq b\}$,

Reflexive.

Reflexivity and Irreflexivity

A relation R on A is **irreflexive** if for every element $a \in A$, $(a, a) \notin R$.

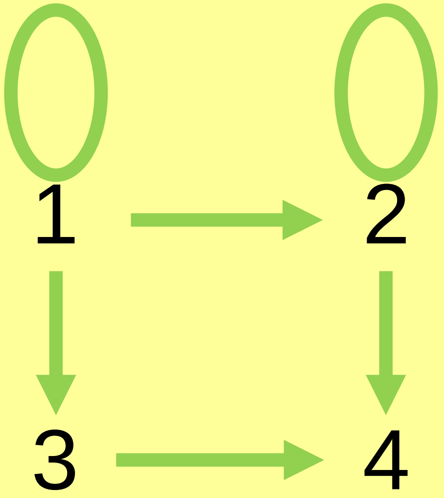
Note: "irreflexive" ≠ "not reflexive".

e.g. If $A = \{1, 2\}$, $R = \{(1, 2), (2, 1), (1, 1)\}$ is

not reflexive because $(2, 2) \notin R$,

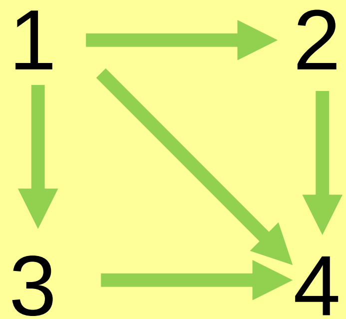
not irrflexive because $(1, 1) \in R$.

Example

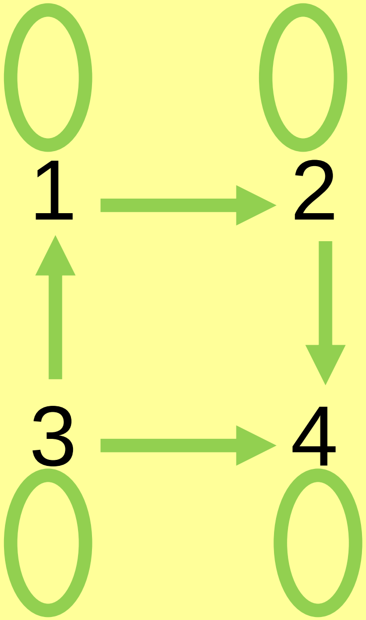


Not Reflexive and Not Irreflexive

Examples



Irreflexive



Reflexive

2. Symmetry and Antisymmetry

- A binary relation R on A is **symmetric** if

$$(a, b) \in R \leftrightarrow (b, a) \in R, \text{ where } a, b \in A.$$

- A binary relation R on A is **antisymmetric** if

$$(a, b) \in R \rightarrow (b, a) \notin R.$$

That is, if $(a, b) \in R \land (b, a) \in R \rightarrow a = b$.

Examples

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a = b\}$

Symmetric , antisymmetric.

$R_2 = \{(a, b) \mid a > b\}$,

Not symmetric, antisymmetric.

$R_3 = \{(a, b) \mid a = b + 1\}$,

Not symmetric, antisymmetric.

Examples

Let $A = \{1, 2, 3\}$.

$R_1 = \{(1, 2), (2, 2), (3, 1), (1, 3)\}$	Not reflexive, not irreflexive, not symmetric, not antisymmetric
$R_2 = \{(2, 2), (1, 3), (3, 2)\}$	Not reflexive, not irreflexive, not symmetric, antisymmetric
$R_3 = \{(1, 1), (2, 2), (3, 3)\}$	Reflexive, not irreflexive, symmetric, antisymmetric
$R_4 = \{(2, 3)\}$	Not reflexive, irreflexive, not symmetric, antisymmetric

3. Transitivity

- A relation R is said to be **transitive** if and only if (for all a, b, c),

$$(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R.$$

e.g. Let $A = \{1, 2\}$.

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is transitive.

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$ is not transitive, $(2, 2) \notin R_2$.

$R_3 = \{(3, 4)\}$ is transitive.

Combining Relations

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$,

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$,

$R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$, then

$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (1, 4)\}$

$R_1 \cap R_2 = \{(1, 1)\}$

$R_1 - R_2 = \{(2, 2), (3, 3)\}$

Composite Relations

- If (a, c) is in R_1 and (c, b) is in R_2 then (a, b) is in $R_2 \circ R_1$.

e.g. R is the relation from {1, 2, 3} to {1, 2, 3, 4}

R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}.

S is the relation from {1, 2, 3, 4} to {0, 1, 2}

S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}.

$S \circ R$ = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}.

8.3 Representing Relations

- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

Using Zero-One Matrices

To represent a relation R by a matrix $M_R = [m_{ij}]$, let $m_{ij} = 1$ if $(a_i, b_j) \in R$, otherwise 0.

Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $R : A \to B$ such that:

$R = \{(2, 1), (3, 1), (3, 2)\}$ then the matrix for R is:

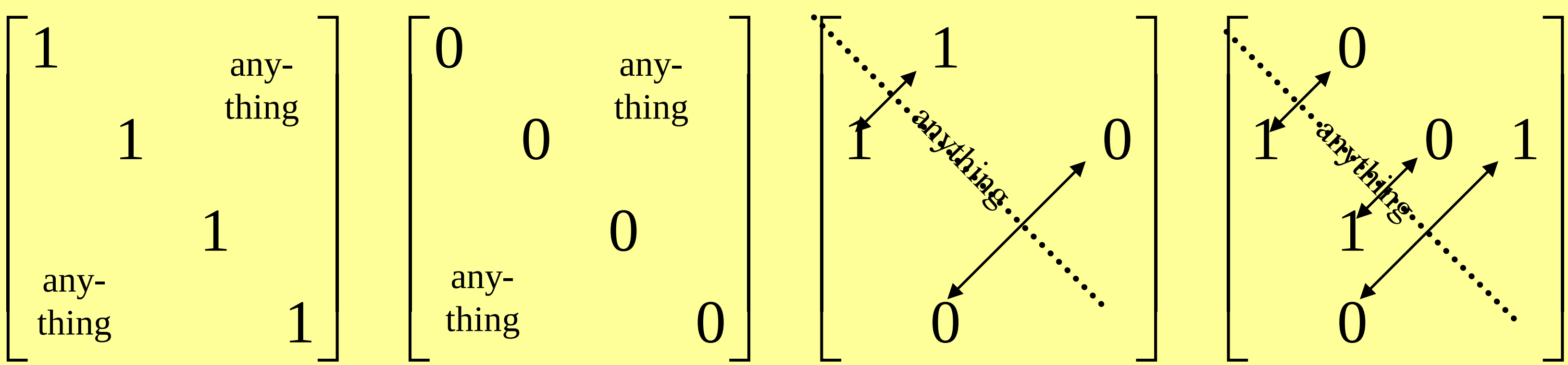
	1	2
1	0	0
2	1	0
3	1	1

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Zero-One Reflexive, Symmetric

The terms: Reflexive, non-reflexive, irreflexive, symmetric and antisymmetric.

- These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal	Irreflexive: all 0's on diagonal	Symmetric: all identical across diagonal	Antisymmetric: all 1's are across from 0's

Example

Is *R* reflexive, symmetric, antisymmetric?

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Reflexive, symmetric, not antisymmetric

Operations

1- Union and the Intersection

The Boolean Operations join $\vee$ and meet $\wedge$ can be used to find the matrices representing the union and the intersection of two relations

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Example

Suppose R_1 and R_2 are relations on a set A which are represented by the matrices:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Operations

2- Composite

Suppose that $R : A \leftrightarrow B$, $S : B \leftrightarrow C$

(Boolean Product)

$$M_{S \circ R} = M_R \Theta M_S$$

Example

Let

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix of $S \circ R$?

$$M_{S \circ R} = M_R \Theta M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Directed Graphs

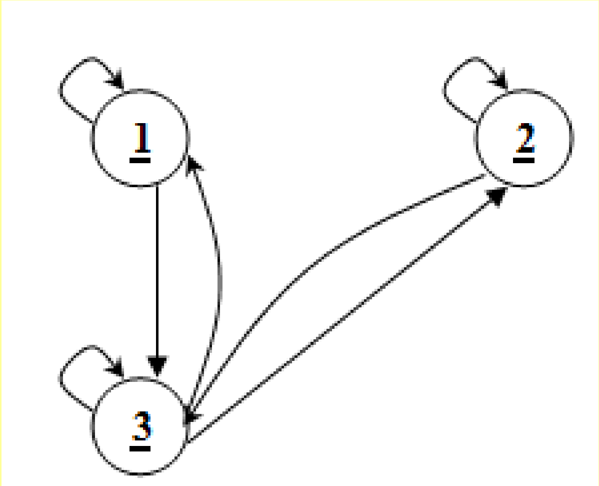
Reflexive (bcz: (1,1), (2,3),(3,3) $\in$ R)

Symmetric ((2,3) ,(3,2) R , (1,3),(3,1) $\in$ R)

Not Ant symmetric ((2,3) ,(3,2) $\in$ R)

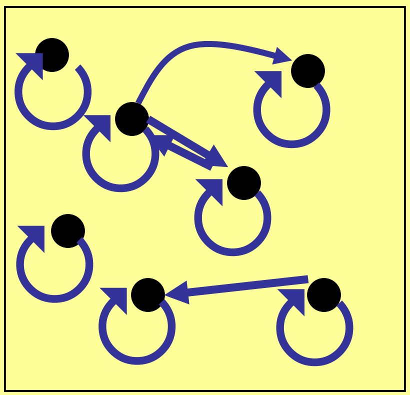
Not Transitive ((1,3) ,(3,2) $\in$ R

but (1,2) $\in$ R)

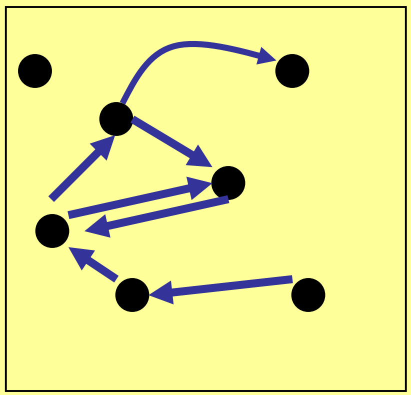


Digraph Reflexive, Symmetric

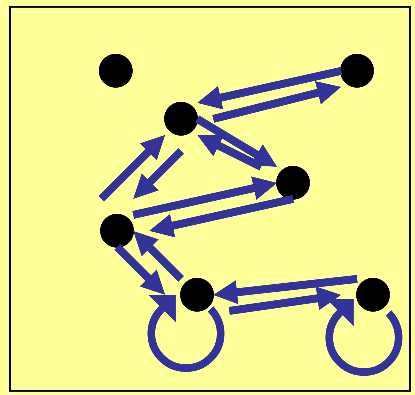
It is extremely easy to recognize the reflexive, irreflexive, symmetric, antisymmetric properties by graph inspection.



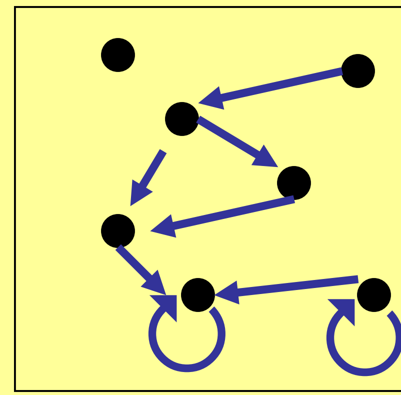
Reflexive:
Every node has a self-loop



Irreflexive:
No node links to itself



Symmetric:
Every link is bidirectional



Antisymmetric:
No link is bidirectional

Not symmetric, non-antisymmetric

Non-reflexive, non-irreflexive